Lecture 5: September 11

Hodge structures and Hodge filtrations. Let me quickly elaborate on the question that came up last time, namely to what extent one can describe Hodge structures in terms of the Hodge filtration. Suppose that

$$H = \bigoplus_{p+q=n} H^{p,q}$$

is a Hodge structure of weight n. The corresponding Hodge filtration consists of the subspaces

$$F^p H = \bigoplus_{i \geq p} H^{i,n-i}.$$

We saw last time that the Hodge filtration appears naturally when we look at families of compact Kähler manifolds (and variations of Hodge structure). In general, it is not possible to recover the Hodge decomposition from the filtration. Some extra data is needed:

(1) Traditionally, one works with $\mathbb{R}$-Hodge structures. Here $H = \mathbb{C} \otimes_{\mathbb{R}} H_{\mathbb{R}}$, where $H_{\mathbb{R}}$ is an $\mathbb{R}$-vector space, and the Hodge structure is assumed to satisfy
$$\overline{H^{p,q}} = H^{q,p}.$$
It is then not hard to show that
$$H^{p,q} = F^p H \cap \overline{F^q H};$$
indeed, the intersection consists exactly of the subspaces $H^{i,j}$ with $i \geq p$ and $j \geq q$.

(2) More importantly, one can also recover the Hodge decomposition with the help of a polarization h. Recall that the Hodge decomposition is orthogonal with respect to h, and that $(-1)^p S$ is positive definite on the subspace $H^{p,q}$. You should convince yourself that
$$H^{p,q} = F^p H \cap (F^{p+1} H)^{\perp},$$
where the orthogonal complement is taken with respect to h. Note that p is the only thing that appears on the right-hand side; to find the value of $q = n - p$, we also need to know the weight of the Hodge structure. In other words, to completely describe a polarized Hodge structure, we need to give the Hodge filtration, the polarization, and the weight n.

Variations of Hodge structure. Let B be a complex manifold of dimension one. Let me briefly recall the main definition from last time; as someone pointed out, I also forgot to say that the polarization needs to be $\mathscr{O}_B$-linear in the first argument.

Definition 5.1. Let $n \in \mathbb{Z}$. A *polarized variation of Hodge structure of weight n* on B consists of the following data:

(a) A holomorphic vector bundle $\mathscr{V}$ and a connection $\nabla \colon \mathscr{V} \to \Omega^1_B \otimes_{\mathscr{O}_B} \mathscr{V}$.
(b) A decreasing filtration by holomorphic subbundles $F^p \mathscr{V}$, called the *Hodge filtration*, such that $\nabla(F^p \mathscr{V}) \subseteq \Omega^1_B \otimes_{\mathscr{O}_B} F^{p-1} \mathscr{V}$.
(c) A hermitian pairing $h \colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}^{\infty}_B$, that is $\mathscr{O}_B$-linear in its first argument, and "flat", in the sense that $dh(s', s'') = h(\nabla s', s'') + h(s', \nabla s'')$ for all local sections $s', s'' \in \mathscr{V}$.

The requirement is that, for every point $b \in B$, the fiber V_b, together with the induced pairing $S_b \colon V_b \otimes_{\mathbb{C}} \overline{V_b} \to \mathbb{C}$ and the filtration by the subspaces $F^p V_b \subseteq V_b$, must come from a Hodge structure of weight n, in the sense we just talked about.

Since the pairing h is hermitian, we get both

$$h(fs', s'') = fh(s', s'') \quad \text{and} \quad h(s', fs'') = \overline{f}h(s', s'')$$

for local sections $s', s'' \in \mathscr{V}$ and $f \in \mathscr{O}_B$. I do not know of a good way to indicate this kind of "sesquilinearity" in the tensor product.

We showed last time that the k-th cohomology groups of a family of compact Kähler manifolds form a polarized variation of Hodge structure of weight k. Another nice class of examples are bundles with a flat metric.

Example 5.2. Let $\mathscr{V}$ be a holomorphic vector bundle on B, and suppose that $\mathscr{V}$ has a *positive definite* hermitian metric $h\colon \mathscr{V} \otimes_{\mathbb{C}} \overline{\mathscr{V}} \to \mathscr{C}_B^\infty$ whose Chern connection is flat. Such bundles correspond exactly to unitary representations of the fundamental group. We can consider $\mathscr{V}$ and h as being a polarized variation of Hodge structure of weight 0, in the following way.

Every fiber of $\mathscr{V}$ can be considered as a polarized Hodge structure of weight 0, with the trivial Hodge decomposition $V_b = V_b^{0,0}$, because h_b is by assumption positive definite on all of V_b. We therefore define the Hodge filtration as

$$F^p\mathscr{V} = \begin{cases} \mathscr{V} & \text{if } p \geq 0, \\ 0 & \text{if } p < 0. \end{cases}$$

Finally, since the curvature of the Chern connection vanishes, the $(1,0)$-part of the Chern connection gives us a holomorphic connection

$$\nabla\colon \mathscr{V} \to \Omega_B^1 \otimes_{\mathscr{O}_B} \mathscr{V}$$

with the property that $dh(s', s'') = h(\nabla s', s'') + h(s', \nabla s'')$. So we obtain all the data we need to have a polarized variation of Hodge structure of weight 0. This class of examples is very useful as a test case: many results about polarized variations of Hodge structure are still nontrivial here, but usually become much easier to prove.

Period domains. Polarized variation of Hodge structure give rise to holomorphic mappings into so-called "period domains", which are basically moduli spaces of polarized Hodge structures. Some of their most interesting properties are easier to describe in that setting.

Let V be a finite-dimensional complex vector space, and $h\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ a hermitian pairing. Our goal is to describe all Hodge structures on V of weight n, with a given set of Hodge numbers, that are polarized by h. One possibility would be to remember all the subspaces in the Hodge decomposition, viewed as points in suitable Grassmannians. But since the Hodge decomposition does not vary holomorphically in families, a variation of Hodge structure would then *not* lead to a holomorphic mapping into the moduli space. For that reason, it is better to work with the Hodge filtration instead.

To simplify the notation, we are going to denote filtrations on V by a single letter such as F; this is really an abbreviation for the collection of subspaces F^pV. Fix one Hodge structure

$$V = \bigoplus_{p+q=n} V_0^{p,q}$$

that is polarized by h, and denote the corresponding Hodge filtration by F_0. The set of all filtrations F such that

$$\dim F^p = \dim F_0^p \quad \text{for all } p \in \mathbb{Z}$$

is a partial flag manifold: a closed submanifold of the product of Grassmannians

$$\prod_{p\in\mathbb{Z}} \operatorname{Gr}(V, \dim F_0^p),$$

defined by the (closed) condition that $F^{p+1} \subseteq F^p$ for all $p \in \mathbb{Z}$. In fact, it is even a smooth projective variety. The set of all filtrations is traditionall denoted by $\check{D}$ and is called the "compact dual" of the period domain.

The condition that a filtration $F \in \check{D}$ comes from a polarized Hodge structure of weight n on V is open. As we saw at the beginning of the lecture, the condition is that $F^p = F^p \cap (F^{p+1})^\perp \oplus F^{p+1}$ for every $p \in \mathbb{Z}$, and that $(-1)^p h$ must be positive definite on the subspace $F^p \cap (F^{p+1})^\perp$ for every $p \in \mathbb{Z}$. Both of these are clearly open conditions. If we let $D \subseteq \check{D}$ be the subset of those filtrations that correspond to polarized Hodge structures of weight n, then D is an open subset of the complex manifold $\check{D}$, and therefore again a complex manifold.

Definition 5.3. The complex manifold D is called the *period domain* for the polarized Hodge structures in question.

To be specific, every point $F \in D$ determines a Hodge structure

$$V = \bigoplus_{p+q=n} V_F^{p,q}$$

of weight n on V that is polarized by the hermitian pairing h. The p-th subspace in the Hodge decomposition is $V_F^{p,n-p} = F^p \cap (F^{p+1})^\perp$, and its dimension is the fixed integer $\dim F_0^p - \dim F_0^{p+1}$.

Example 5.4. A simple but instructive example are polarized Hodge structures of elliptic curves. In Lecture 2, we saw that the Hodge structure

$$H^1(E, \mathbb{C}) = H^{1,0}(E) \oplus H^{0,1}(E)$$

is polarized by the pairing $(2\pi i)^{-1} \int \alpha \wedge \overline{\beta}$, which is negative definite on $H^{1,0}(E)$ and positive definite on $H^{0,1}(E)$. Abstractly, we are therefore looking for Hodge structures on $V = \mathbb{C}^2$ that are polarized by the hermitian pairing

$$h\big((x', y'), (x'', y'')\big) = x'\overline{x''} - y'\overline{y''}.$$

This is the standard way to write a hermitian pairing of signature $(1,1)$. The Hodge filtration is determined by the one-dimensional subspace F^1, and so $\check{D} = \mathbb{P}^1$. What is D in this case? In order for a the subspace

$$F^1 = \mathbb{C} \cdot (x, y)$$

to correspond to a polarized Hodge structure of weight 1, we need

$$0 < (-1)^1 h\big((x, y), (x, y)\big) = |y|^2 - |x|^2,$$

or equivalently, $|y| > |x|$. In particular, $y \neq 0$, and so after rescaling, $F^1 = \mathbb{C} \cdot (t, 1)$, where $|t| < 1$. This shows that the period domain

$$D = \big\{ t \in \mathbb{C} \bigm| |t| < 1 \big\}$$

is isomorphic to the unit disk in $\mathbb{C}$. You should check that the Hodge structure corresponding to a point $t \in D$ is

$$\mathbb{C} \cdot (t, 1) \oplus \mathbb{C} \cdot (1, \bar{t}).$$

It the literature, the period domain for Hodge structures of elliptic curves is usually taken to be the upper half-plane (which is of course isomorphic to the unit disk); this is more natural if one is trying to parametrize the elliptic curves as quotients $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$, but less natural from the point of view of Hodge structures.

Both D and $\check{D}$ are also homogeneous spaces. This is very useful in practice. The Hodge filtration F_0 that we chose at the beginning can serve as a base point in $D \subseteq \check{D}$. Any other filtration $F \in \check{D}$ can be obtained from F_0 by applying an element of $\mathrm{GL}(V)$: indeed, we can choose a basis in V that is adapted to the filtration F_0, and another basis that is adapted to the filtration F, and then consider the linear transformation that takes the first basis to the second. In other words, $\mathrm{GL}(V)$ acts transitively on $\check{D}$, and therefore

$$\check{D} \cong \mathrm{GL}(V)/B,$$

where $B \subseteq \mathrm{GL}(V)$ is the stabilizer of F_0. Note that $\mathrm{GL}(V)$ is a complex Lie group, and so we get another explanation for the fact that $\check{D}$ is a complex manifold.

On the period domain D, we only have an action by the smaller group

$$G_{\mathbb{R}} = \left\{ g \in \mathrm{GL}(V) \;\middle|\; h(gv', gv'') = h(v', v'') \text{ for all } v', v'' \in V \right\}.$$

Note that $G_{\mathbb{R}}$ is only a real Lie group, due to the fact that h is conjugate-linear in the second argument. It takes a little bit more work to see that the action is again transitive. Recall that the Hodge decomposition

$$V = \bigoplus_{p+q=n} V_0^{p,q}$$

is orthogonal with respect to h, and that $(-1)^p h$ is positive definite on $V_0^{p,q}$. We can therefore choose an h-orthogonal basis for each subspace $V_0^{p,q}$, such that the h-length of every vector is $(-1)^p$. If we choose the same kind of basis for a second Hodge filtration $F \in D$, the linear transformation that takes the first basis to the second basis will then belong to $G_{\mathbb{R}}$. Consequently,

$$D \cong G_{\mathbb{R}}/H,$$

where $H \subseteq G_{\mathbb{R}}$ is again the stabilizer of the reference Hodge filtration F_0. Note that this way of writing D as a homogeneous space does not remember the fact that D is actually a complex manifold (because $G_{\mathbb{R}}$ is not a complex Lie group).

Lemma 5.5. *The group H is compact.*

Proof. Let $C_0 \in \mathrm{End}(V)$ be the operator that acts as multiplication by $(-1)^p$ on the subspace $V_0^{p,q}$. Since h is a polarization,

$$\langle v', v'' \rangle_0 = h(C_0 v', v'')$$

is a positive definite inner product on V. The unitary group

$$U = U\big(V, \langle\,\rangle_0\big) = \left\{ g \in \mathrm{GL}(V) \;\middle|\; \langle gv', gv'' \rangle_0 = \langle v', v'' \rangle_0 \text{ for all } v', v'' \in V \right\}$$

is of course a compact group. If an element $g \in G_{\mathbb{R}}$ fixes F_0, then it also fixes the Hodge decomposition, because $V_0^{p,q} = F_0^p \cap (F_0^{p+1})^{\perp}$. But then g preserves the inner product $\langle\,\rangle_0$, and so

$$H \subseteq U \cap B.$$

It follows that H is a compact group. You can check for yourself that one actually has equality here, meaning that $H = U \cap B$. $\square$

The Lie algebra of $\mathrm{GL}(V)$ is of course $\mathrm{End}(V)$; the Lie algebra of $G_{\mathbb{R}}$ is the smaller real Lie algebra

$$\mathfrak{g} = \left\{ A \in \mathrm{End}(V) \;\middle|\; h(Av', v'') + h(v', Av'') = 0 \text{ for all } v', v'' \in V \right\}$$

of all endomorphisms that are skew-adjoint with respect to h.

Period mappings. Suppose now that we have a polarized variation of Hodge structure of weight n on a one-dimensional complex manifold B, with flat bundle $(\mathscr{V}, \nabla)$, Hodge bundles $F^p\mathscr{V}$, and polarization $h_{\mathscr{V}}$. This is a family of Hodge structures, but all the Hodge structures are on different vector spaces, whereas the period domain parametrizes Hodge structures on a *fixed* vector space. In order to have a "moduli map", we therefore need to find a trivialization for the vector bundle $\mathscr{V}$. Locally on B, and in fact on every simply-connected open subset, we can always find a trivialization by ∇-flat sections. If we want a global trivialization, we need to go to the universal covering space

$$\pi \colon \tilde{B} \to B.$$

Define $\tilde{\mathscr{V}} = \pi^*\mathscr{V}$, with the induced connection $\tilde{\nabla} = \pi^*\nabla$ and the induced pairing $\tilde{h}_{\mathscr{V}} = \pi^* h_{\mathscr{V}}$. Now there is a clever way to avoid choosing a base point: let

$$V = \left\{ v \in H^0(\tilde{B}, \tilde{\mathscr{V}}) \;\middle|\; \tilde{\nabla} v = 0 \right\}$$

be the space of all flat sections of the pullback of $\mathscr{V}$. Since $\tilde{B}$ is simply connected, restriction from V to the fiber of $\tilde{\mathscr{V}}$ at any point of $\tilde{B}$ is an isomorphism, which means that V is a model for the general fiber of $\mathscr{V}$, without actually choosing any points. In particular, we have trivialization

$$V \otimes_{\mathbb{C}} \mathscr{O}_{\tilde{B}} \cong \tilde{\mathscr{V}},$$

and so we can consider the Hodge bundles $F^p\tilde{\mathscr{V}} = \pi^* F^p\mathscr{V}$ as being subbundles of the trivial bundle with fiber V. By the universal property of the Grassmannian, we therefore obtain a holomorphic mapping

$$\Phi \colon \tilde{B} \to \check{D},$$

where $\check{D}$ parametrizes all filtrations F on the vector space V such that $\dim F^p = \operatorname{rk} F^p\mathscr{V}$ for all $p \in \mathbb{Z}$.

The vector space V also inherits a hermitian pairing

$$h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C},$$

as follows. Since $\tilde{h}_{\mathscr{V}}$ is a flat pairing, we have

$$d\tilde{h}_{\mathscr{V}}(v', v'') = \tilde{h}_{\mathscr{V}}(\tilde{\nabla} v', v'') + \tilde{h}_{\mathscr{V}}(v', \nabla v'') = 0$$

for every $v', v'' \in V$, and so the C^∞-function $\tilde{h}_{\mathscr{V}}(v', v'')$ is constant on $\tilde{B}$. We define $h(v', v'')$ to be that constant value. Since we started from a polarized variation of Hodge structure of weight n, all Hodge structures are polarized by the pairing h, and so the image of Φ is contained in the period domain $D \subseteq \check{D}$. The holomorphic mapping

$$\Phi \colon \tilde{B} \to D$$

is called the *period mapping* of the polarized variation of Hodge structure. (In order to get something defined on B, one has to quotient D by the action of the monodromy group, but this construction will not be important for us.)

Example 5.6. Period mappings have some very interesting (but not obvious) properties. Here is the simplest example of this. Consider a polarized variation of Hodge structure of weight 1, coming from a family of elliptic curves. In this case, the period mapping $\Phi \colon \tilde{B} \to D$ goes into the unit disk. By the classification of simply connected Riemann surfaces, $\tilde{B}$ is either $\mathbb{P}^1$, or $\mathbb{C}$, or the unit disk D. In the first two cases, the period mapping is constant, which means that the variation of Hodge structure is locally trivial. In the third case, the period mapping is a holomorphic mapping from the unit disk to itself, and by the Schwarz-Pick lemma, we know for example that

$$|\Phi(t)| \leq |t|$$

for every $t \in D$, and that equality for any $t \in D$ implies that Φ must be a rotation. We will see later that, altough a general period domain does not look at all like the unit disk, period mappings nevertheless have similar properties as in this example.

Tangent spaces. Since $\check{D} = \mathrm{GL}(V)/B$, the tangent space to $\check{D}$ (and hence D) at the reference point F_0 can be identified with

$$\mathrm{End}(V)/\left\{ A \in \mathrm{End}(V) \;\middle|\; A(F_0^p) \subseteq F_0^p \text{ for all } p \in \mathbb{Z} \right\}.$$

Our goal is to construct a hermitian metric on D that behaves in some ways like a the negatively curved metric on the unit disk (which is responsible for the Schwarz-Pick lemma). This metric is ultimately induced by the polarization on the reference Hodge structure. Let me start with some general remarks.

Let V be a complex vector space, h a hermitian pairing, and suppose that

$$V = \bigoplus_{p+q=n} V^{p,q}$$

has a Hodge structure of weight n polarized by h. Then

$$E = \mathrm{End}(V) \cong V^* \otimes_{\mathbb{C}} V$$

inherits a Hodge structure of weight $-n+n=0$, which we write as

$$E = \bigoplus_{\ell \in \mathbb{Z}} E^{\ell,-\ell}.$$

We can describe the Hodge decomposition very concretely as follows:

$$E^{\ell,-\ell} = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(V^{p,q}) \subseteq V^{p+\ell,q-\ell} \text{ for every } p,q \in \mathbb{Z} \right\}$$

Roughly speaking, we are decomposing the space of matrices into blocks, corresponding to the given decomposition of V. Likewise,

$$F^p E = \left\{ A \in \mathrm{End}(V) \;\middle|\; A(F^i V) \subseteq F^{i+p} V \text{ for every } i \in \mathbb{Z} \right\}.$$

In particular, the space $F^0 E$ consists exactly of those endomorphisms that map the Hodge filtration on V into itself.

Interestingly, the Hodge structure on E is actually an $\mathbb{R}$-Hodge structure. This is due to the pairing. The pairing h is nondegenerate, and so every $A \in \mathrm{End}(V)$ has an adjoint $A^* \in \mathrm{End}(V)$, defined by the condition that

$$h(Av', v'') = h(v', A^* v'')$$

for all $v', v'' \in V$. The mapping $A \mapsto A^*$ is a conjugate-linear involution of $\mathrm{End}(V)$, due to the fact that $(\lambda A)^* = \overline{\lambda} A^*$ for every $\lambda \in \mathbb{C}$. The set

$$E_{\mathbb{R}} = \left\{ A \in \mathrm{End}(V) \;\middle|\; A = A^* \right\}$$

of all self-adjoint endomorphisms is therefore a real subspace of E with the property that $E = \mathbb{C} \otimes_{\mathbb{R}} E_{\mathbb{R}}$. It is not hard to see that $A \in E^{\ell,-\ell}$ if and only if $A^* \in E^{-\ell,\ell}$, and so we actually have an $\mathbb{R}$-Hodge structure on $E_{\mathbb{R}}$.

The polarization on V is going to induce a polarization on E, which can again be described in the following concrete manner.

Lemma 5.7. *The pairing*

$$E \otimes_{\mathbb{C}} \overline{E} \to \mathbb{C}, \quad (A, B) \mapsto \mathrm{tr}(A \circ B^*),$$

polarizes the Hodge structure on E.

Proof. I only had time to show that the Hodge decomposition is orthogonal with respect to the trace pairing. Take $A \in E^{\ell,-\ell}$ and $B \in E^{k,-k}$. Then $B^* \in E^{-k,k}$, and therefore $A \circ B^* \in E^{\ell-k,k-\ell}$. If $k \neq \ell$, then $A \circ B^*$ shifts the Hodge decomposition either up or down in a nontrivial way, and so some power of $A \circ B^*$ must vanish. But this means that $A \circ B^*$ is nilpotent, and so $\mathrm{tr}(A \circ B^*) = 0$. $\square$